Industry-Level Capital-Labour Isoquants

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Abstract

In both theoretical and applied economics, the long-run conditions of production for an industry are often presented in the form of a capital-labour isoquant. This representation cannot be derived from the full long-run system of industry-level production functions specified in terms of the physical inputs.

Introduction

It is not unusual, in either theoretical or applied economics, to find that the conditions of production in a particular industry are represented by a capital-labour isoquant, $k(l)$**.** This is especially so, of course, in the context of long-run equilibrium where all inputs are variable. The object of this short paper is to examine how such a representation of technical possibilities is related to the fuller, physical production function, representation.(More exactly, we shall employ the cost function approach and, indeed, since we assume constant-return-to-scale, the unit cost function approach.)

The Framework

Consider an economy with n single-product industries, exhibiting constant returns and employing only circulating capital. Let p_j be the price of the jth commodity (which is uniform throughout the economy), while w and r are the wage rate and the rate of interest (both also uniform throughout). If $c_j(\quad)$ is the unit cost function for industry j then, in long-run equilibrium,

$$p_j = c_j[(1+r)p_1, (1+r)p_2, ..., (1+r)p_n, w] \qquad (1)$$

if wages are paid *ex-post* and all produced inputs are paid for *ex-ante*. We suppose that system (1) can be solved to give (when $w > 0$)

$$p_j = w f_j(1+r) \qquad (2)$$

Labour use per unit of output in industry j will of course be given by

$$l_j = \frac{\partial c_j}{\partial w}(\quad) \qquad (3)$$

where the right- hand side of (3) is homogeneous of degree zero in the input prices. From (2), then, l_j can be written as a function of $(1+r)$ alone and, indeed, if we take l_j to be a Hicksian-substitute for every other input used in industry j then l_j will be a monotonically-increasing function of $(1+r)$.

Similarly, each produced input use per unit of output in industry j, a_{ij}, will be given by the appropriate partial derivative of $c_j(\quad)$, will be homogeneous of degree zero in the input prices and can be written as a function of $(1+r)$ alone. The capital-output ratio in industry j will, of course, be given by

$$k_j = \Sigma_i a_{ij}(p_i / p_j)$$

and, taking account of (2), we see that k_j also can be written as a function of $(1+r)$ alone.

Consider now all the alternative possible long-run equilibria corresponding to all the non-negative values of *r* that permit a non-negative real wage. At each such value of r, both l_j and k_j are determined and hence as r increases from zero, a (k_j / l_j) locus is traced out. There is no doubt, then, that the full system (1) does imply a $k_j(l_j)$ relation for each $j = 1, ..., n.$

We cannot leap straight to the conclusion that there is a proper capital-labour isoquant for each industry, however. In order to qualify as such an isoquant, our $k_j(l_j)$ relation must have at least two properties; it must slope downwards and it must be convex from above. Moreover, if a capital-labour isoquant is to be of use in economic analysis it must have the property that it is tangential to the 'input iso-cost-line', i.e., that $(dk_j / dl_j) = -[w/(1+r)p_j]$. The following sections show that our $k_j(l_j)$ relations need not satisfy these three requirements.

First and Second Derivatives

The $k_j(l_j)$ relation can fail to be downward-sloping and/or everywhere convex from above even when the unit cost functions - the $c_j(\ \ )$ - are of the simplest, most stereotypical marginalist kind. Quite specifically, we shall consider a two-industry economy $(n = 2)$ in which the first industry has a constant-elasticity-of-substitution unit cost function (with $\sigma < 1$), while the second industry has a Cobb-Douglas unit cost function. Each industry uses as a produced input only the product of the other industry.

To begin, let us set aside consideration of the second industry for the moment and write down for the first industry the price-cost relation

$$p_1^{(1-\sigma)} = \kappa_1[(1+r)p_2]^{(1-\sigma)} + \lambda_1 w^{(1-\sigma)} \qquad (4)$$

, where $\kappa_1 + \lambda_1 = 1 > \sigma.$ From (4),

$$a_{21} = \kappa_1[\frac{p_1}{(1+r)p_2}]^{\sigma}$$

and hence

$$k_1 = \frac{\kappa_1}{(1+r)^{\sigma}}(\frac{p_2}{p_1})^{(1-\sigma)} \qquad (5)$$

If we choose our measurement units so that $p_1 = p_2 = w$ when $r = 0,$ we see from (5) that $k_1 = \kappa_1$ when $r = 0.$ From (5), however, $k_1 = \kappa_1$ *whenever*

$$(p_2 / p_1)^{(1-\sigma)} = (1+r)^{\sigma}$$

, i.e., from (4), *whenever*

$$(\frac{w}{p_1}) = [\frac{1-\kappa_1(1+r)}{\lambda_1}]^{(\frac{1}{1-\sigma})} \qquad (6)$$

It is important to be clear that (6) is *not* the economy's real wage-interest rate frontier; it is simply a condition the satisfaction of which is equivalent to $k_1 = \kappa_1$; it depends *solely* on the cost function for industry 1. The real wage-interest rate frontier, by contrast, depends on *both* $c_1(\quad)$ and $c_2(\quad)$. It is to be expected, then, that $c_2(\quad)$ can be so chosen that that frontier intersects (6) not only at $r = 0,$ but also at some $r > 0;$ it would follow at once that $k_j(r)$ is non-monotonic and hence that $k_j(l_j)$ is also non-monotonic (given our assumption that $l_j(r)$ is increasing).

To see how easily that expectation can be fulfilled, let us return to the assumption that $c_2(\quad)$ is of the Cobb-Douglas from and write, specifically,

$$p_2 = [(1+r)p_1]^{1/2} w^{1/2} \tag{7}$$

From (4), (5) and (7),

$$1 = (1+r)k_1 + \left(\frac{\lambda_1}{\kappa_1^{\;2}}\right)\frac{k_1^{\;2}}{(1+r)^{(1-3\sigma)}} \tag{8}$$

Now (8) is clearly satisfied by $r = 0$ and $k_1 = \kappa_1$; but it is also satisfied *whenever* $k_1 = \kappa_1$ and

$$\kappa_1(1+r) + \frac{\lambda_1}{(1+r)^{(1-3\sigma)}} = \kappa_1 + \lambda_1 \tag{9}$$

It is easy to show that (9) has a solution $r_* > 0$ if and only if

$$3\sigma < \left(\frac{1-2\kappa_1}{1-\kappa_1}\right) \tag{10}$$

(implying, of course, that $\kappa_1 < 1/2$). Whenever (10) holds in (4), the $k_1(l_1)$ relation is *increasing* at $r = 0$, and is as shown in Figure 1. Note that the 'problem' with the $k_1(l_1)$ relation arises here for *low* values of r, i.e., for relevant values of the interest rate.

Figure 1 Here

(Before continuing our discussion of the case $n = 2$, we may turn aside to suppose that (4) still holds in industry 1 but that there are $(n-1)$ other industries. Equation (6) will still apply and it should be possible to choose $c_j(\;)$ functions for the $(n-1)$ other industries in such a way that the economy's real wage-interest rate frontier intersects (6) many times; that is, so that $k_1 = \kappa_1$ for many positive rates of interest and thus $k_1(l_1)$ is decidedly non-monotonic. Leaving the reader to pursue this possibility, we now return to our simple $n = 2$ example.)

If inequality (10) is reversed then $k_1(l_1)$ is downward-sloping for all relevant r – but this is not, of course, a sufficient condition to make $k_1(l_1)$ a conventional capital-labour isoquant. Some tedious differentiation and manipulation shows that $(d^2k_1 / dl_1^2) > 0$ will hold good (at $r = 0$) only if *another* inequality (of the form $3\sigma > f(\kappa_1)$) holds. When it does not, we have Figure 2, in which $k_1(l_1)$ has the conventional negative slope but not everywhere the conventional curvature. Note that it is again at *low* values of r that the convention is violated.

Figure 2 Here

To conclude this section we present Figure 3, which shows how different combinations of σ and κ_1 yield different signs for (dk_1 / dl_1) and (d^2k_1 / dl_1^2), *evaluated at* $r = 0$*;* note that Figure 3 is only sketched somewhat approximately and that $0.81 < \kappa_1* < 0.82$. In region A, $(dk_1 / dl_1) > 0 > (d^2k_1 / dl_1^2)$; in region B, $(dk_1 / dl_1) < 0 > (d^2k_1 / dl_1^2)$; while in region C, $(dk_1 / dl_1) < 0 < (d^2k_1 / dl_1^2)$,the conventional case.

Figure 3 Here

The Tangency Condition

A downward-sloping, convex from above $k_j(l_j)$ relation is only useful in economic analysis if it has the property that its (absolute) slope is equal to the ratio between the real wage rate and $(1+r)$. Now in long-run equilibrium

$$1 = w_j l_j + (1+r)k_j \qquad (11)$$

where w_j is the real wage measured in terms of commodity j. On suppressing the j subscripts we may rewrite (11) in differential form:

$$0 = [wdl + (1+r)dk] + (ldw + kdr) \qquad (12)$$

and we see from (12) that $-(dk/dl) = (w/1+r)$ if and only if $-(dw/dr) = (k/l)$. Now it is well known that, at $r = 0$, $-(dw/dr) = (k^v/l^v)$, where k^v and l^v are the capital-output and labour-output ratios for the *vertically integrated* sector j. Thus, at $r = 0$, $-(dk/dl) = (w/1+r)$ if and only if $(k/l) = (k^v/l^v)$. This condition will hold for any particular industry only by a complete fluke. And it could be satisfied for every industry only if all the (k_j/l_j) were exactly equal! Thus outside the fantasy world of an 'as if 'one-commodity system, in which relative commodity prices never change as r varies, at least two industries must *fail* to satisfy the $-(dk/dl) = (w/1+r)$ condition at $r = 0$. Even if their $k_j(l_j)$ relations do happen to be downward-sloping and convex from above, they do not constitute useful capital-labour isoquants.

Concluding Remarks

When one considers a long-run equilibrium system of n industries, as defined in (1) above, one finds that variation of the interest rate does imply the existence of a $k_j(l_j)$ relation for each industry. Such relations cannot (generically) be interpreted as meaningful capital-labour isoquants, however. They can slope upwards, they need not be globally convex from above and they need not have an absolute slope equal to the relevant ‘input-price-ratio’. Industry-level capital-labour isoquants provide an entirely spurious representation of long-run production possibilities and they should therefore be cast out of the economist’s toolbox.