Fixed Capital in the Corn-Tractor Model

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ABSTRACT

When fixed capital use is introduced into the simple corn-tractor model, capital reversing can occur even with equal proportions and triple-switching of techniques is possible.

1. INTRODUCTION

Debates in capital theory have made extensive use of the 'corn-tractor' model, in which any given technique of production involves the use of labour and a technique-specific type of machine to produce either further machines of the same kind or the consumption commodity. Both defenders and critics of marginalist capital theory employed this model. However, the 'defenders' often assumed radioactive depreciation, whilst the 'critics' generally (and with good reason) avoided this assumption and supposed capital to be circulating capital. This was wise on the critics' part in so far as the radioactive depreciation treatment of fixed capital is not a responsible approach to fixed capital theory; as Morishima once pointed out, it involves supposing that some fraction of a Roman centurion's knife is still at work today – and with unchanged efficiency! This paper considers how the corn-tractor analysis works out when the machine used in a given industry is assumed to work with constant efficiency for a given number of production periods. It is not implied, of course, that this assumption represents 'the truth' about fixed capital use

– only that it is a lot more sensible than the radioactive depreciation assumption.

While the ‘critics’ did well to avoid the last-named assumption, it is perhaps rather surprising that they did so by assuming circulating capital, since Sraffa (1960, p.70) had referred to ‘the remarkable effect that with any rise in the rate of profits the value of [a balanced stock of machines] *rises* relative to the original value of a new machine’ (and Sraffa made little use of such words as ‘remarkable’). Be that as it may, the remarkable effect will be central to what follows, as we try to show how a sensible treatment of fixed capital can be decidedly damaging for familiar marginalist results, even within the confines of the simple corn-tractor model.[1] We shall not attempt a complete analysis (see below) but will attempt to provide a systematic basis for such an analysis.[2]

2. SOME PRELIMINARIES

When circulating capital is assumed, the familiar basic equations for the corn-tractor model are, in Hicksian notation,

$$p = bw + (1+r)ap$$

$$1 = \beta w + (1+r)\alpha p$$

where the consumption commodity is the numéraire, p is the price of the producer good, w the wage rate (paid *ex post*) and r is the rate of interest.

Notes:

1. Sraffian authors have, of course, provided splendid papers on general, abstract fixed capital models but not nearly enough work has been done, it might be thought, on simpler models.

2. Some non-systematic forays into the fixed-capital field may be found in Hodgson and Steedman (1977) and Steedman (1979, 1994, 2006, 2011, 2013)

But if the producer good works (with constant efficiency) for n periods in the producer good industry and for *v* periods in the consumer good industry, used machines being non-transferable between industries, then we must change these familiar equations to

$$p = bw + apc_n(r) \qquad (1)$$

$$1 = \beta w + \alpha pc_v(r) \qquad (2)$$

where

$$c_n(r) \equiv \frac{r(1+r)^n}{(1+r)^n - 1} \qquad (3)$$

Note that it is no longer necessary to assume, as in the familiar circulating capital model, that a<1; now we need only assume that a<n. Note also that $c_n(r)$ is inversely related to n and that both $(\partial c / \partial r)$ and $(\partial^2 c / \partial r^2)$ are positive.

From (1) and (2)

$$p = \frac{b}{\beta + \alpha bc_v(r) - a\beta c_n(r)} \qquad (4)$$

$$w = \frac{1 - ac_n(r)}{\beta + \alpha bc_v(r) - a\beta c_n(r)} \qquad (5)$$

and it is immediately clear that the nature of the $p(r)$ and $w(r)$ relationships will now depend not only on how (α / β) compares with (a/b) but also on how v compares with n. (There is no presumption that $v = n$; after all, the machine is being used differently in the two industries.) We shall return to these comparisons in the next section but first we note some useful properties of (3).

It follows from (3) that

(i) $c_n(r) \equiv \frac{(1+r)^n}{1+(1+r)+...+(1+r)^{n-1}}$

(ii) $c_{n+1}(r) \equiv \frac{(1+r)c_n(r)}{1+c_n(r)}$

(iii) $c_{2n}(r) \equiv \frac{(1+r)^n c_n(r)}{1+(1+r)^n}$

(iv) $c_n(r) \equiv (\frac{1}{n}) + (\frac{n+1}{2n})r + (\frac{n^2-1}{12n})r^2 + r^3(....)$, which can be useful when r is small.

(v) If $v < n$ then $1 < \frac{c_v(r)}{c_n(r)} < (\frac{n}{v})$

3. RELATIVE CAPITAL INTENSITY

As is well-known, in the circulating capital model $(v = n = 1)$ it matters whether (α / β) exceeds, equals, or is less than (a/b). If $(v = n \geq 2)$ this may still matter – but suppose that $v \neq n$. Now, it might seem more relevant to compare $(\alpha / \beta v)$ with (a/bn) and we shall, indeed, see below that this comparison has its importance. We note, however, that even if (α / β) equals (a/b), the comparison of v with n is still relevant to 'relative capital-intensity' if we think of this in terms of comparing $[\alpha c_v(r)/\beta]$ with $[ac_n(r)/b]$. What, then, is the best definition here of relative capital intensity?

Rather than engage in potentially logomatic discussion of this question, we proceed as follows. It is well known that marginalist theory tends to work out more smoothly when consumer-good production is not less capital-intensive than producer-good production. So we shall assume for the most part *both* that $(\alpha/\beta) \geq (a/b)$ *and* that $v \leq n$, since each assumption works to make the consumption commodity industry the more capital-intensive one. (We assume, unless otherwise stated, that $2 \leq v \leq n$ and that n is finite.)

More specifically, the next section will consider the case $\alpha b = a\beta$ and $v = n$; section 5 will deal with the case $\alpha b > a\beta$ and $v = n$; and section 6 will consider the case $\alpha b = a\beta$ and $v < n$. Section 7 will be somewhat more general.

4. THE CASE $\alpha b = a\beta$ and $v = n$

We have here 'equal proportions' in the clearest possible sense and relations (4) and (5) simplify to

$$\beta p = b \qquad (4')$$

$$\beta w = 1 - a c_n(r) \qquad (5')$$

As we see from (4′), p is a constant, independent of r. But this by no means implies that the downward-sloping $w(r)$ frontier in (5′) is a straight line. To the contrary, it is necessarily convex from below, which means that the capital-output ratio, (k/y), increases with r. That this is so, despite the constant p, is a manifestation of the Champernowne-Kahn effect (1953-54) that Sraffa found 'remarkable'.

The fact that a very simple model, involving equal proportions in a very strong sense, must give rise to a wage-interest rate frontier that is not a straight line will already signal, to any reader conversant with capital theory, that the presence of fixed capital can be highly problematic for marginalist theory. To reinforce this signal, let us consider an economy with many techniques of the kind considered in this section, for each of which $v = n = 2$. When we compare any two of these alternative techniques, we find that $\Delta a . \Delta\beta < 0$ and we can be sure that *no* reswitching can occur; as r increases across any switchpoint, in fact, 'a' will fall and 'β' will rise. As the interest rate increases from zero, the capital-output ratio, (k/y), will always increase whilst the technique is unchanged (see above) and will always fall as the choice of technique is changed across a switchpoint. Overall, the $(k/y)(r)$ relation will be of a 'saw-tooth' form and certainly not a monotonically falling relation of the traditional marginalist form.

What happens if we now allow there to be infinitely many alternatives (of the kind in question)? There are various possible outcomes and we simply illustrate with one example. Let the economy $w(r)$ frontier – the outer envelope of the individual technique $w(r)$ frontiers – be

$$w = 1 - 4r$$

For this to be the outer envelope, it must always be true that

$$1 - \frac{a(1+r)^2}{(2+r)} = \beta(1-4r)$$

and

$$\frac{a(1+r)(3+r)}{(2+r)^2} = 4\beta$$

, so that each convex-from-below individual frontier is tangential to the economy frontier. It follows easily that

$$a = \frac{4(2+r)^2}{(1+r)(11+r)}$$

$$\beta = (\frac{3+r}{11+r}) \qquad \text{, for } 0 \le r \le 0.25,$$

and that

$$a = \frac{(9\beta - 1)^2}{4(5\beta - 1)} \qquad \text{, for } (3/11) \le \beta \le (13/45).$$

It is straightforward to show, for this economy with infinitely many *equal-proportions* techniques, that $(k/y) = 4$ when $r = 0$, falls to a minimum at $r \doteq 11.65\%$ and then *rises* back to 4 when $r = 25\%$. In the presence of fixed capital, the capital-output ratio can be a non-monotonic function of the interest rate even with infinitely many equal-proportions techniques, all involving the same length of machine life. So much for surrogate production functions; Samuelson needed not only his notorious $(\alpha / \beta = a/b)$ assumption but *also* his highly dubious radioactive depreciation assumption. (As the reader may easily check, $[k + (dw/dr)] < 0,$ whenever r is positive, so that $[(dy/dk) - r]$ has the same sign as (dk/dr). Only exceptionally will (dy/dk) be equal to the interest rate.)

Thus far, reswitching has featured in this section only by its absence. Yet it can arise even under our present assumptions, provided only that the two equal-proportions techniques have *different* values of $v = n$. To see this, specify two (r_j, w_j) points, with $0 < r_1 < r_2$ and $w_1 > w_2 > 0$. For any given $v = n, c_n(r_1)$ and $c_n(r_2)$ are known and (5′) now provides two simple equations determining (a, β). Now, keeping the r_j

and w_j unchanged, *change* $v = n$ and determine a new pair (a, β). We have shown that, by construction, two techniques – say, $(a_1, \beta_1; v_1 = n_1)$ and $(a_2, \beta_2; v_2 = n_2)$ - generate $w(r)$ frontiers passing through both (r_1, w_1) and (r_2, w_2). Hence two techniques, each exhibiting equal proportions in the strongest sense, can double-switch. It simply is not true that the assumption of ubiquitous equal proportions is sufficient to rule out reswitching. (The reader might wish to construct an example in which a $(v = n = 2)$ technique is used at both 'low' and 'high' values of r, while a $(v = n = 3)$ technique is used at 'intermediate' values of r. Note that, as r *increases* across the first switchpoint, the change is to a technique using a *longer*-lived machine. If this is found surprising, note that the surprising result occurs here at the *low* switchpoint r, whilst capital-reversing and so on are generally associated with subsequent switchpoints.

In brief, in the presence of fixed capital, both capital-reversing and reswitching are perfectly compatible with equal-proportions techniques and can be highly disruptive for familiar marginalist claims. We now turn to consider what may occur when the machine life is the same in both industries but the machine-labour use ratios differ.

5. THE CASE $\alpha b > a\beta$ and $v = n$

In this case (4) and (5) become

$$p = \frac{b}{\beta + (\alpha b - a\beta)\, c_n(r)} \qquad (4'')$$

$$w = \frac{1 - a c_n(r)}{\beta + (\alpha b - a\beta) c_n(r)} \qquad (5'')$$

, where $(\alpha b - a\beta) > 0$. Clearly p always falls as r increases. Now we know from the Champernowne-Kahn effect that the value of the aggregate capital stock will always increase with r when expressed in terms of a new machine as numéraire – but how will it vary with r *when expressed in terms of the consumption commodity*? Since $p(r)$ is decreasing, the answer is not immediately obvious. Nor is it very attractive for a general r – but it *is* reasonably simple at $r = 0$, so we focus here on that case.

On differentiating the $r.h.s.$ of (5 '') twice with respect to r and then utilizing property (iv) of $c_n(r)$ – see Section 2 above – one finds that $\ddot{w}(0) > 0$ iff

$$(\frac{\alpha b}{a\beta}) > 1 + \frac{n(n-1)}{2(n+2)a} > \frac{3(n+1)}{2(n+2)} \qquad (6)$$

In the familiar case of circulating capital $(n = v = 1)$, (6) simply reminds us that $\ddot{w}(0)$ is positive or negative, and hence the capital-output ratio at $r = 0$ is decreasing or increasing in r, according as αb is greater than or less than $a\beta$. But when $n = v > 1$, $\alpha b > a\beta$, is *not sufficient* to ensure that the capital-output ratio is decreasing with r (at $r = 0$); the Champernowne-Kahn effect may still more than offset the falling $p(r)$ effect. Thus there may or may not be capital-reversing at $r = 0$.

The $w(r)$ frontier (5 ''), then, can be convex from below or convex from above at $r = 0$, in the present case. What is its shape for $0 < r \le R$, where $ac_n(R) = 1$ and thus $w = 0$? It can be shown that $\ddot{w}(R)$ has the same sign as

$$[2\dot{c}_n^2(R) - c_n(R)\ddot{c}_n(R)]\alpha b - 2\dot{c}_n^2(R)a\beta \qquad (7)$$

, where the term within square brackets is always positive. If $(\alpha b / a\beta)$ is 'small' then we see from (7) that $\ddot{w}(R) < 0$ but if $(\alpha b / a\beta)$ is sufficiently greater than unity then $\ddot{w}(R) > 0$ is possible. More broadly, if $(\alpha b / a\beta)$ is sufficiently small then the $w(r)$ frontier will be everywhere convex from below, while if it is sufficiently large then that frontier will be everywhere $(0 \le r \le R)$ convex from above. For 'intermediate' values of $(\alpha b / a\beta)$, however, it can be that $w(r)$ is convex from below for small r but convex from above for small w; that is, the $w(r)$ frontier has an inflexion point within the positive quadrant. Suppose, for example, that $v = n = 2$ and that $(\frac{\alpha b}{\beta}) = a + 0.2$. It can then be shown that $\ddot{w}(0) < 0$ and that $\ddot{w}(r) = 0$ for r just under 10%. Provided only that $10\% < R$, we have an example of a $w(r)$ frontier that is convex from below to begin with but convex from above as r approaches R. In the present case, then, there are three qualitatively different possible shapes for the $w(r)$ frontier.

Consider a frontier with an inflexion point in the positive quadrant. It will at once be clear that such a frontier can be cut three times, in the positive quadrant, by a downward sloping straight line. Let there be two techniques, both with the $v = n$ property. The first is a circulating capital technique, with $v = n = 1$, and $\alpha b = a\beta$; its $w(r)$ frontier will be a straight line. The second technique has $n = v > 1$ and $\alpha b > a\beta$; let its $w(r)$ frontier have a relevant inflexion point. The two techniques, each with the property $v = n$, can exhibit *triple*-switching. In the presence of fixed capital, even when $v = n$ is imposed, there is the possibility not merely of double-switching but of *triple*-switching. Clearly the capital-output ratio will *not* be a monotonic, decreasing function of the interest rate! Moreover, the length of life of the machine used will be, as r increases, first $n = 1$, then $n = 2$, then $n = 1$ again and finally $n = 2$. Note that the life *increases* with r across the first switch. (Of course, we are referring to the lives of qualitatively different machines here.) At the second switch-point there will be consumption-reversal, the phenomenon that, of all the possibilities revealed by the capital theory debates, was found the most disturbing by such authors as Samuelson and Burmeister.

In brief, the case $\alpha b > a\beta$ and $v = n$ can yield decidedly unconventional results.

6. THE CASE $\alpha b = a\beta$ and $v < n$

Here, (4) and (5) become

$$\beta p = \frac{b}{1 + a[c_v(r) - c_n(r)]} \qquad (4''')$$

$$\beta w = \frac{1 - a c_n(r)}{1 + a[c_v(r) - c_n(r)]} \qquad (5''')$$

From (4''') we see at once that $p < (a/\alpha) = (b/\beta)$ always holds and that $\dot{p}(0) < 0$. Since the Champernowne-Kahn effect is always positive, it is not immediately clear whether there will be capital-reversing at $r = 0$. It is clear, however, that $p(r)$ cannot be monotonic, since $[c_v(r) - c_n(r)]$ tends to zero both as r approaches -1 and as r increases without limit (with $v > 1$). To illustrate the point, consider the case $v = 3 < 6 = n$; it is easily seen that $[c_3(r) - c_6(r)]$ is the reciprocal of

$$[(1+r)^2+(1+r)+1+(1+r)^{-1}+(1+r)^{-2}+(1+r)^{-3}] \qquad (8)$$

and that (8) has a unique minimum at r just over 19%. Thus when $v=3$, $n=6$ and $R>20\%$, say, $p(r)$ falls for $0 \le r < 19\%$ (approximately) and then rises for $19\% < r \le R$ (approximately). We can thus be sure that, in this example, as r approaches R from below, the capital-output ratio will be rising and hence the $w(r)$ frontier will be convex from below.

More generally, despite the ambiguity concerning the initial direction of change in the price of a new machine, it can be shown that the capital-output ratio always rises at first, in the present case. There are now only two possible qualitative shapes for the wage-interest rate frontier, therefore. *If* there is a point of inflexion, there will again be the possibility of *triple*-switching. Consumption-reversal will still occur across the second switch point.

In this case, then, quite unconventional behaviour can again occur, even though $\alpha b = a\beta$.

7. A MORE GENERAL CASE.

We now allow $(\alpha b - a\beta)$ to be positive, negative or zero but maintain the assumption that $n > v$. Starting from the wage-interest rate frontier (5), from the familiar national accounting identity $y \equiv w + rk$ and from the property (iv) of $c_n(r)$ given in Section 2, one can deduce, after a little algebra, that the capital-output ratio will be increasing with r, at $r=0$, iff:

$$\left(\frac{\alpha / \beta v}{a / bn}\right) < \frac{n(v+1)(3n-v+4)+n(n^2-v^2)+(n^2/a)(v^2-1)+a(n-v)(2n-v+3)}{2n(v+1)(v+2)+a(n-v)(2v-n+3)} \qquad (9)$$

On the *l.h.s.* of (9) we have a plausible measure of the machine-labour ratio in the consumption sector relative to that in the machine sector. If $n=v$ then (9) naturally reduces to (6) above – but what can be said more generally? If the *l.h.s.* of (9) is unity then (9) *always* holds good for $n \geq v$. Indeed the same is true provided that the *l.h.s.* is less than or equal to $(n+1/v+1)$; this is so because this condition ensures that $p(r)$ is non-decreasing at $r=0$. If $\alpha b = a\beta$ then (9) holds good for $n \geq v \geq 2$. Going beyond these special cases, (9) is clearly not an especially pleasant relationship to work with. One possibility would be to take (a, n) as given (implying that R is taken as given), and then to consider the relation between $(\alpha b / \beta)$ on the *l.h.s.* and the resulting function of v on the *r.h.s.*. For now, we leave these matters to the interest of the reader.

8. CONCLUDING REMARKS

The assumption that fixed capital works with constant efficiency throughout a given lifetime is certainly a restrictive assumption but it is at least far more respectable than the lazy, and rather silly, assumption of radioactive depreciation. We have found that, even in the simple context of the corn-tractor model, it can give rise to phenomena inconsistent with simple marginalist economic theory. Capital-reversing can easily occur, as can triple-switching; nor is there any basis for linking shorter machine lives with higher interest rates. We therefore urge Sraffa-inspired authors to pay more attention to the analysis of fixed capital in simple models of production and hope that enough has been said here to provide a systematic basis for such further analysis.

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